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GUANGSHI LÜ, JIE WU & WENGUANG ZHAI

Abstract. In this paper we study the mean square of the error term $\Delta_k^*(Q,x)$ in a divisor problem related to the Epstein zeta-function. An asymptotic formula has been obtained when $k=2$.

1. Introduction

This is the third part of our series of papers on a divisor problem related to the Epstein zeta-function [10, 11]. First we recall some notation there. Let $\ell \geqslant 2$, $\mathbf{y} := (y_1, \dots, y_\ell)$ and $\mathbf{A} = (a_{ij})$ be an integral matrix such that $a_{ii} \equiv 0 \,(\mathrm{mod}\, 2)$ for $0 \leqslant i \leqslant \ell$. Thus a positive definite quadratic form $Q(\mathbf{y})$ can be written as

$$Q(\mathbf{y}) = \frac{1}{2}\mathbf{y}^{\mathrm{t}}\mathbf{A}\mathbf{y} = \sum_{1\leqslant i<j\leqslant \ell} a_{ij}y_iy_j + \frac{1}{2}\sum_{1\leqslant i\leqslant \ell} a_{ii}y_i^2,$$

where $\mathbf{y}^{\mathrm{t}}$ is the transpose of $\mathbf{y}$. The corresponding Epstein zeta-function is initially defined by the Dirichlet series

$$Z_Q(s) := \sum_{\mathbf{y}\in\mathbb{Z}^\ell\smallsetminus\{\mathbf{0}\}} Q(\mathbf{y})^{-s} = \sum_{n\geqslant 1} a_n n^{-s} \quad (\Re e\, s > \ell/2), \tag{1.1}$$

where a_n is the number of the solutions of the equation $Q(\mathbf{y}) = n$ with $\mathbf{y} \in \mathbb{Z}^\ell$. It is known that $Z_Q(s)$ has an analytic continuation to the whole complex plane $\mathbb{C}$ with only a simple pole at $s = \ell/2$, and satisfies a functional equation of Riemann type (cf. [13]). For each integer $k \geqslant 1$, we define $a_k(n)$ by

$$Z_Q(s)^k = \sum_{n\geqslant 1} a_k(n)n^{-s} \quad (\Re e\, s > \ell/2) \tag{1.2}$$

and put

$$\Delta_k^*(Q,x) := \sum_{n\leqslant x} a_k(n) - x^{\ell/2}P_k(\log x), \tag{1.3}$$

where $P_k(\log x) := x^{-\ell/2}\,\mathrm{Res}_{s=\ell/2}(Z_Q(s)^k x^s s^{-1})$ is a polynomial of $\log x$ of degree $k-1$. The study on asymptotic behavior of the error term $\Delta_k^*(Q,x)$ has received

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much attention [8, 1, 13]. In particular Sankaranarayanan [13] showed that for $k \geqslant 2$ and $\ell \geqslant 3$,

$$\Delta_k^*(Q, x) \ll x^{\ell/2-1/k+\varepsilon}, \tag{1.4}$$

where and throughout this paper ε denotes an arbitrarily small positive constant. Recently inspired by Iwaniec's book [6], Lü [10] marked that (1.4) can been improved for the quadratic forms of level one (see [6, Chapter 11]). These quadratic forms are defined by $Q(\mathbf{y}) = \frac{1}{2}\mathbf{y}^{\mathrm{t}}\mathbf{A}\mathbf{y}$ verifying the following supplementary conditions:

$$\ell \equiv 0 \,(\bmod 8), \qquad \mathbf{A} \text{ is equivalent to } \mathbf{A}^{-1}, \qquad \det(\mathbf{A}) = 1.$$

Denote by $\mathcal{Q}_\ell$ the set of such quadratic forms. For $Q \in \mathcal{Q}_\ell$, we have [6, (11.32)]

$$a_n = A_\ell \sigma_{\ell/2-1}(n) + a_f(n, Q) \quad (n \geqslant 1),$$

where

$$A_\ell := \frac{(2\pi)^{\ell/2}}{\zeta(\ell/2)\Gamma(\ell/2)}, \qquad \sigma_k(n) = \sum_{d|n} d^k,$$

$\zeta(s)$ is the Riemann zeta-function, $\Gamma(s)$ is the Gamma function and $a_f(n, Q)$ is the nth Fourier coefficient of a cusp form $f(z, Q)$ of weight $\ell/2$ with respect to the full modular group $\mathrm{SL}(2, \mathbb{Z})$. Thus

$$Z_Q(s) = A_\ell \zeta(s)\zeta(s - \ell/2 + 1) + L(s, f) \qquad (\Re e\, s > \ell/2), \tag{1.5}$$

where $L(s, f)$ is the Hecke L-function associated with $f(z, Q)$. According to Deligne's well known work [2], we know

$$|a_f(n, Q)| \leqslant n^{(\ell/2-1)/2}\tau(n) \quad (n \geqslant 1), \tag{1.6}$$

where $\tau(n)$ is the divisor function. With the help of these properties, Lü [10] (for $k \geqslant 4$) and Lü, Wu & Zhai [11] (for $k = 2, 3$) obtained

$$\Delta_k^*(Q, x) \ll x^{\ell/2-1+\theta_k+\varepsilon},$$

where θ_k is the exponent in the classical k-dimension divisor problem

$$\Delta_k(x) := \sum_{n \leqslant x} \tau_k(n) - \operatorname*{Res}_{s=1}(\zeta(s)^k x^s s^{-1}) \ll x^{\theta_k+\varepsilon} \qquad (x \geqslant 2).$$

In particular we can take $\theta_2 = 131/416$ [4], $\theta_3 = 43/96$ [7] and $\theta_k = (k-1)/(k+2)$ for $k \geqslant 4$ [15]. Besides, an Ω-result has been established in [11]: if $8 \mid \ell$ and $Q(\mathbf{y}) \in \mathcal{Q}_\ell$, then we have for $k = 2, 3$ that

$$\Delta_k^*(Q, x) = \Omega\big(x^{\ell/2-1+(k-1)/2k}(\log x)^{(k-1)/(2k)}(\log_2 x)^a(\log_3 x)^{-b'}\big),$$

where $a = \frac{k+1}{2k}(k^{(2k)/(k+1)} - 1)$, b' is any constant greater than $\frac{3k-1}{4k}$ and $\log_r$ denotes the r-fold iterated logarithm.

The aim of this paper is to study the mean square of $\Delta_k^*(Q, x)$.

Theorem 1. *If $8 \mid \ell$, then for any quadratic form $Q(\mathbf{y}) \in \mathcal{Q}_\ell$, we have*

$$\int_1^T |\Delta_2^*(Q, x)|^2 \,\mathrm{d}x = C_\ell T^{\ell-1/2} + O\big(T^{\ell-1}(\log T)^3 \log_2 T\big),$$

where

$$g_a(n) := \sum_{d|n} \frac{\tau(d)\tau(n/d)}{d^a}, \qquad C_\ell := \frac{3A_\ell^4}{(2\ell-1)\pi^2} \sum_{n=1}^{\infty} \frac{g_{(\ell-3)/2}(n)^2}{n^{3/2}}. \tag{1.7}$$

The estimate $O\big(T^{\ell-1}(\log T)^3 \log_2 T\big)$ follows from the result of [9] on the mean square of $\Delta_2(x)$.

Theorem 2. *For $k \geqslant 2$, $8 \mid \ell$ and $Q(\mathbf{y}) \in \mathcal{Q}_\ell$, we define*

$$\beta_k := \inf\left\{b_k : \int_1^T |\Delta_k(x)|^2 \,\mathrm{d}x \ll T^{1+2b_k+\varepsilon}\right\},$$

$$\beta_k^* := \inf\left\{b_k^* : \int_1^T |\Delta_k^*(Q,x)|^2 \,\mathrm{d}x \ll T^{\ell-1+2b_k^*+\varepsilon}\right\}.$$

Then $\beta_k^* = \beta_k$. Further we have $\beta_k^* \geqslant (k-1)/2k$ and the equality holds if the Lindelöf hypothesis of $\zeta(s)$ is true.

Ivić [5,] proved that

$$\beta_3 = 1/3, \quad \beta_4 = 3/8, \quad \beta_5 \leqslant 119/260, \quad \beta_6 \leqslant 1/2, \quad \beta_7 \leqslant 39/70.$$

According to Theorem 2, the same estimates for β_k^* hold.

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2. An Expression of $\Delta_2^*(Q,x)$

In [11], we actually established the formula

$$\Delta_2^*(Q,x) = A_\ell^2 x^{\ell/2-1} \sum_{d \leqslant x} \frac{\tau(d)}{d^{\ell/2-1}} \Delta_2\left(\frac{x}{d}\right) + O(x^{\ell/2-1+\varepsilon}).$$

From it we can deduce Ω-result of $\Delta_2^*(Q,x)$. However, it is not enough to prove Theorem 1. So first we will give a better expression of $\Delta_2^*(Q,x)$.

Lemma 2.1. *If $8 \mid \ell$, then for any quadratic form $Q(\mathbf{y}) \in \mathcal{Q}_\ell$, we have*

$$\Delta_2^*(Q,x) = A_\ell^2 x^{\ell/2-1} \sum_{d \leqslant x} \frac{\tau(d)}{d^{\ell/2-1}} \left(\Delta_2\left(\frac{x}{d}\right) - \frac{1}{4}\right) - 2A_\ell x^{\ell/2-1} \sum_{d \leqslant x} \frac{b(d)}{d^{\ell/2-1}} \psi\left(\frac{x}{d}\right) + O\big(x^{\ell/2-5/4}\big),$$

where $\psi(t) := \{t\} - \frac{1}{2}$ and $\{t\}$ denotes the fractional part of t.

Proof. From (1.5) we have

$$Z_Q(s)^2 = A_\ell^2 \zeta(s)^2 \zeta(s-\ell/2+1)^2 + A_\ell \zeta(s)\zeta(s-\ell/2+1)L(s,f) + L(s,f)^2.$$

Here the last two terms do not appear when $\ell = 8, 16$ since there are no cusp forms of weights 4 and 8 with respect to $\mathrm{SL}(2,\mathbb{Z})$. Thus we can write

$$\begin{aligned}\sum_{n\leqslant x} a_2(n) &= A_\ell^2 \sum_{d\leqslant x} \tau(d) \sum_{m\leqslant x/d} \tau(m) m^{\ell/2-1} \\ &\quad + 2A_\ell \sum_{d\leqslant x} b(d) \sum_{m\leqslant x/d} m^{\ell/2-1} + \sum_{d\leqslant x} c(d),\end{aligned} \tag{2.1}$$

where $b(n)$ and $c(n)$ are defined by

$$\zeta(s)L(s,f) = \sum_{n=1}^\infty b(n)n^{-s} \qquad \text{and} \qquad L(s,f)^2 = \sum_{n=1}^\infty c(n)n^{-s}$$

for $\Re e s > \ell/2$, respectively. By using Deligne's bound (1.6), it is easy to see that

$$|b(n)| \leqslant n^{(\ell/2-1)/2}\tau_3(n) \qquad \text{and} \qquad |c(n)| \leqslant n^{(\ell/2-1)/2}\tau_4(n). \tag{2.2}$$

Thus

$$\sum_{n\leqslant x} (|b(n)| + |c(n)|) \ll x^{\ell/4+1/2}(\log x)^3. \tag{2.3}$$

By partial summation we have

$$\begin{aligned}\sum_{m\leqslant x} \tau(m)m^{\ell/2-1} &= \frac{2}{\ell}x^{\ell/2}\Big(\log x - \frac{2}{\ell} + 2\gamma\Big) + x^{\ell/2-1}\Delta_2(x) \\ &\quad - (\ell/2-1)\int_1^x \Delta_2(t)t^{\ell/2-2}\mathrm{d}t.\end{aligned}$$

By using Voronoï's well known formula [16]:

$$\int_1^t \Delta_2(u)\,\mathrm{d}u = \frac{t}{4} + O(t^{3/4}),$$

a simple partial summation leads to

$$(\ell/2-1)\int_1^x \Delta_2(t)t^{\ell/2-2}\mathrm{d}t = \frac{1}{4}x^{\ell/2-1} + O\big(x^{\ell/2-5/4}\big).$$

Combining these, we find that

$$\begin{aligned}\sum_{m\leqslant x} \tau(m)m^{\ell/2-1} &= \frac{2}{\ell}x^{\ell/2}\Big(\log x - \frac{2}{\ell} + 2\gamma\Big) \\ &\quad + x^{\ell/2-1}\bigg(\Delta_2(x) - \frac{1}{4}\bigg) + O\big(x^{\ell/2-5/4}\big).\end{aligned} \tag{2.4}$$

Similarly (even easier)

$$\sum_{m\leqslant x} m^{\ell/2-1} = \frac{2}{\ell}x^{\ell/2} - x^{\ell/2-1}\psi(x) + O\big(x^{\ell/2-2}\big). \tag{2.5}$$

Now the required result follows from (2.1), (2.3), (2.4) and (2.5). □

3. Proof of Theorem 1

3.1. **Beginning of the Proof.** Let

$$\widetilde{\Delta}_2^*(Q,x) := \frac{\Delta_2^*(Q,x)}{A_\ell^2 x^{\ell/2-1}} \qquad \text{and} \qquad \widetilde{C}_\ell := \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{g_{(\ell-3)/2}(n)^2}{n^{3/2}}.$$

Clearly it is sufficient to prove that

$$\int_1^T \big|\widetilde{\Delta}_2^*(Q,x)\big|^2\,\mathrm{d}x = \frac{\widetilde{C}_\ell}{6}T^{3/2} + O\big(T(\log T)^3\log_2 T\big). \tag{3.1}$$

According to Lemma 2.1, we can write

$$\widetilde{\Delta}_2^*(Q,x) = U(x) - V(x) + O\big(x^{\ell/2-5/4}\big),$$

where

$$U(x) := \sum_{d\leqslant x}\frac{\tau(d)}{d^{\ell/2-1}}\bigg(\Delta_2\Big(\frac{x}{d}\Big) - \frac14\bigg), \qquad V(x) := \frac{2}{A_\ell}\sum_{d\leqslant x}\frac{b(d)}{d^{\ell/2-1}}\psi\Big(\frac{x}{d}\Big).$$

Next we shall prove

$$\int_1^T U^2(x)\mathrm{d}x = \frac{\widetilde{C}_\ell}{6}T^{3/2} + O\big(T(\log T)^3\log_2 T\big), \tag{3.2}$$

$$\int_1^T U(x)V(x)\mathrm{d}x \ll T(\log T)^2, \tag{3.3}$$

which imply (3.1).

3.2. **Preparation.** In this subsection, we shall prove some preliminary estimates, which are useful later.

Lemma 3.1. *Let $a>0, b>1, \ell>a+b$ and $A\geqslant 1$. We have*

$$\sum_{\substack{d_1,d_2\leqslant T\\ m_1,m_2\leqslant M\\ d_1m_2=d_2m_1}}\frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1d_2)^{\ell/2-a/2}(m_1m_2)^{b/2}} = \sum_{n=1}^{\infty}\frac{g_{(\ell-a-b)/2}(n)^2}{n^b} + O_A\bigg(\frac{(\log T)^3}{T^{b-1}}\bigg), \tag{3.4}$$

$$\sum_{\substack{d_1,d_2\leqslant T\\ m_1,m_2\leqslant M\\ d_1m_2\neq d_2m_1}}\frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1d_2)^{\ell/2-3/4}(m_1m_2)^{3/4}}\frac{1}{|\sqrt{m_1/d_1}-\sqrt{m_2/d_2}|} \ll_A (\log T)^3\log_2 T, \tag{3.5}$$

$$\sum_{\substack{d_1,d_2\leqslant T\\ m_1,m_2\leqslant M}}\frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1d_2)^{\ell/2-3/4}(m_1m_2)^{3/4}}\frac{1}{\sqrt{m_1/d_1}+\sqrt{m_2/d_2}} \ll_A (\log T)^3\log_2 T. \tag{3.6}$$

uniformly for $1\leqslant T\leqslant M\leqslant T^A$, where $g_r(n)$ is defined as in (1.7).

Proof. First we write

$$
\begin{aligned}
S_1(T,M) &= \sum_{n\leqslant TM} \frac{1}{n^b}\bigg(\sum_{\substack{d\leqslant T;\, m\leqslant M\\ dm=n}} \frac{\tau(d)\tau(m)}{d^{(\ell-a-b)/2}}\bigg)^2\\
&= \sum_{n=1}^{\infty} \frac{g_{(\ell-a-b)/2}(n)^2}{n^b} + O\bigg(\sum_{n>T}\frac{g_{(\ell-a-b)/2}(n)^2}{n^b}\bigg).
\end{aligned} \tag{3.7}
$$

It is easy to see that $g_r(n)$ is multiplicative, $g_r(p) = 2+2/p^r$ and $g_r(p^\nu) \ll_r (\nu+1)$ for all p and $\nu \geqslant 1$. Applying Theorem 2.1 of [14] with $x = y$ and $\kappa = 4$ to $g_r(n)^2$ leads to the following inequality

$$\sum_{n\leqslant x} g_r(n)^2 \ll_r x(\log x)^3 \quad (r>0,\ x\geqslant 2).$$

From it and (3.7), we can easily deduce (3.4).

Similarly we can write

$$S_2(T,M) \leqslant \sum_{\substack{n,n'\leqslant TM\\ n\neq n'}} \frac{g_{\ell/2-2}(n)g_{\ell/2-2}(n')}{(nn')^{3/4}} \frac{1}{|\sqrt{n}-\sqrt{n'}|} \ll_A (\log T)^3 \log_2 T.$$

In the last step we have used the bound of Lau & Tsang [9].

The estimate (3.6) is an immediate consequence of (3.4) with $a=b=2$ and (3.5) if noting that

$$\frac{1}{\sqrt{m_1/d_1}+\sqrt{m_2/d_2}} \ll \begin{cases} \left(\dfrac{d_1d_2}{m_1m_2}\right)^{1/4} & \text{if } m_1/d_1 = m_2/d_2\\[2ex] \dfrac{1}{|\sqrt{m_1/d_1}-\sqrt{m_2/d_2}|} & \text{if } m_1/d_1 \neq m_2/d_2. \end{cases}$$

□

3.3. **Proof of** (3.2)**.** According to Meurman [12], we have

$$\Delta_2(x) - \frac{1}{4} = \frac{x^{1/4}}{\sqrt{2}\pi}\sum_{m\leqslant M}\frac{\tau(m)}{m^{3/4}}\cos\left(4\pi\sqrt{xm}-\frac{\pi}{4}\right) + E(x) \tag{3.8}$$

for all $M > x > 1$, where

$$E(x) \ll \begin{cases} x^{-1/4} & \text{if } \|x\| \geqslant x^{5/2}M^{-1/2},\\ x^{\varepsilon} & \text{if } \|x\| \leqslant x^{5/2}M^{-1/2}.\end{cases} \tag{3.9}$$

Thus we can write, with the choice of $M = T^{10} > x$,

$$U(x) = A(x) + B(x), \tag{3.10}$$

where

$$
\begin{aligned}
A(x) &:= \frac{x^{1/4}}{\sqrt{2}\pi}\sum_{d\leqslant x}\frac{\tau(d)}{d^{\ell/2-3/4}}\sum_{m\leqslant M}\frac{d(m)}{m^{3/4}}\cos\left(4\pi\sqrt{\frac{m}{d}x}-\frac{\pi}{4}\right),\\
B(x) &:= \sum_{d\leqslant x}\frac{\tau(d)}{d^{\ell/2-1}}E\left(\frac{x}{d}\right).
\end{aligned}
$$

In view of the identity $2\cos u\cos v = \cos(u-v)+\cos(u+v)$, we easily see that

$$A(x)^2 = A_1(x)+A_2(x)+A_3(x),$$

where

$$A_1(x) := \frac{x^{1/2}}{4\pi^2}\sum_{\substack{d_1,d_2\leqslant x\\ m_1,m_2\leqslant M\\ m_1d_2=m_2d_1}} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1d_2)^{\ell/2-3/4}(m_1m_2)^{3/4}},$$

$$A_2(x) := \frac{x^{1/2}}{4\pi^2}\sum_{\substack{d_1,d_2\leqslant x\\ m_1,m_2\leqslant M\\ m_1d_2\neq m_2d_1}} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1d_2)^{\ell/2-3/4}(m_1m_2)^{3/4}}\cos\left(4\pi\left(\sqrt{\frac{m_1}{d_1}}-\sqrt{\frac{m_2}{d_2}}\right)\sqrt{x}\right),$$

$$A_3(x) := \frac{x^{1/2}}{4\pi^2}\sum_{\substack{d_1,d_2\leqslant x\\ m_1,m_2\leqslant M}} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1d_2)^{\ell/2-3/4}(m_1m_2)^{3/4}}\cos\left(4\pi\left(\sqrt{\frac{m_1}{d_1}}+\sqrt{\frac{m_2}{d_2}}\right)\sqrt{x}\right).$$

By using (3.4) we have

$$\begin{aligned}\int_1^T A_1(x)\,\mathrm{d}x &= \frac{1}{4\pi^2}\sum_{\substack{d_1,d_2\leqslant T\\ m_1,m_2\leqslant M\\ m_1d_2=m_2d_1}} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1d_2)^{\ell/2-3/4}(m_1m_2)^{3/4}}\int_{\max\{d_1,d_2\}}^T x^{1/2}\,\mathrm{d}x\\ &= \frac{\widetilde{C}_\ell}{6}T^{3/2}+O\big(T(\log T)^3\big).\end{aligned}$$

With the help of the first derivative test and (3.5), we get

$$\begin{aligned}\int_1^T A_2(x)\,\mathrm{d}x &\leqslant \sum_{1\leqslant k\leqslant 2\log T}\left|\int_{T/2^k}^{T/2^{k-1}} A_2(x)\,\mathrm{d}x\right|\\ &\ll \sum_{1\leqslant k\leqslant 2\log T}(T/2^k)S_2(T/2^{k-1},M)\\ &\ll T(\log T)^3\log_2 T.\end{aligned}$$

Similarly we have

$$\int_1^T A_3(x)\mathrm{d}x \ll T.$$

Combining these estimates, we find that

$$\int_1^T A(x)^2\,\mathrm{d}x = \frac{\widetilde{C}_\ell}{6}T^{3/2}+O\big(T(\log T)^3\log_2 T\big). \tag{3.11}$$

By Cauchy's inequality, it follows

$$B(x)^2 \leqslant \sum_{d\leqslant x}\frac{\tau(d)^2}{d^2}\sum_{d\leqslant x}\frac{1}{d^{\ell-4}}E\left(\frac{x}{d}\right)^2 \ll \sum_{d\leqslant x}\frac{1}{d^{\ell-4}}E\left(\frac{x}{d}\right)^2,$$

which combining (3.9) allows us to deduce that

$$
\begin{aligned}
\int_1^T B(x)^2 \,\mathrm{d}x &\ll \sum_{d\leqslant T} \frac{1}{d^{\ell-5}}\bigg(\int_{\substack{1\\ \|t\|\leqslant t^{5/2}M^{-1/2}}}^{T/d} t^{\varepsilon}\,\mathrm{d}t + \int_{\substack{1\\ \|t\|\geqslant t^{5/2}M^{-1/2}}}^{T/d} t^{-1/2}\,\mathrm{d}t\bigg)\\
&\ll \sum_{d\leqslant T} \frac{1}{d^{\ell-5}}\bigg\{\Big(\frac{T}{d}\Big)^{7/2+\varepsilon}\frac{1}{M^{1/2}} + \Big(\frac{T}{d}\Big)^{1/2}\bigg\}\\
&\ll T^{1/2}.
\end{aligned}
\tag{3.12}
$$

From (3.11) and (3.12), we get, via Cauchy's inequality, that

$$
\int_1^T A(x)B(x)\,\mathrm{d}x \ll T. \tag{3.13}
$$

Now the asymptotic formula (3.2) follows from (3.10), (3.11), (3.12) and (3.13).

3.4. **Proof of** (3.3)**.** By using Theorem 4.5 in Graham and Kolesnik [3]

$$
\Delta_2(u) = -2\sum_{m\leqslant\sqrt{u}} \psi(u/m) + O(1)
$$

and (2.2), we have

$$
\int_1^T U(x)V(x)\,\mathrm{d}x \ll \sum_{d\leqslant T}\frac{\tau(d)}{d^{\ell/2-1}} \sum_{m\leqslant (T/d)^{1/2}} \sum_{n\leqslant T} \frac{\tau_3(n)}{n^{\ell/4-1/2}} |I(d,m,n)| + T, \tag{3.14}
$$

where

$$
I(d,m,n) := \int_{\max\{dm^2,n\}}^T \psi\Big(\frac{x}{dm}\Big)\psi\Big(\frac{x}{n}\Big)\,\mathrm{d}x.
$$

For $\psi(u)$, it is well-known that the finite Fourier expansion

$$
\psi(u) = -\sum_{1\leqslant h\leqslant H} \frac{\sin(2\pi hu)}{\pi h} + O\bigg(\min\Big\{1, \frac{1}{H\|u\|}\Big\}\bigg)
$$

holds for any $H\geqslant 2$. It is easily seen that for any $r>0$

$$
\begin{aligned}
\int_{\max\{dm^2,n\}}^T \min\Big\{1, \frac{1}{H\|x/r\|}\Big\}\,\mathrm{d}x &= r\int_{m^2}^{T/r} \min\Big\{1, \frac{1}{H\|t\|}\Big\}\,\mathrm{d}t\\
&\ll T\int_0^{1/2} \min\Big\{1, \frac{1}{Ht}\Big\}\,\mathrm{d}t\\
&\ll TH^{-1}\log H.
\end{aligned}
$$

From these we deduce

$$
I(d,m,n) \ll \sum_{h_1,h_2\leqslant H} \frac{|I(h_1,h_2)|}{h_1h_2} + \frac{T(\log H)^2}{H} \tag{3.15}
$$

where

$$\begin{aligned} I(h_1,h_2) &:= \int_{\max\{dm^2,n\}}^{T} \sin\left(\frac{2\pi h_1 x}{dm}\right)\sin\left(\frac{2\pi h_2 x}{n}\right)\mathrm{d}x \\ &\ll \begin{cases} 1/|h_1/dm - h_2/n| & \text{if } h_1 n \neq h_2 dm, \\ T & \text{if } h_1 n = h_2 dm. \end{cases} \end{aligned}$$

Here we have used the identity $2\sin u \sin v = \cos(u-v) - \cos(u+v)$ and the first derivative test when $h_1 n \neq h_2 dm$.

Inserting (3.15) into (3.14), we get

$$\int_1^T U(x)V(x)\,\mathrm{d}x \ll T S_4(T,H) + S_5(T,H) + T + \frac{T^{3/2}(\log H)^2}{H},$$

where

$$\begin{aligned} S_4(T,H) &:= \sum_{d\leqslant T}\frac{\tau(d)}{d^{\ell/2-1}}\sum_{m\leqslant (T/d)^{1/2}}\sum_{n\leqslant T}\frac{\tau_3(n)}{n^{\ell/4-1/2}}\sum_{\substack{h_1,h_2\leqslant H\\ h_1 n = h_2 dm}}\frac{1}{h_1 h_2} \\ &\leqslant \sum_{r\leqslant HT}\frac{1}{r^2}\sum_{n\mid r}\frac{\tau_3(n)}{n^{\ell/4-3/2}}\sum_{h_2 dm = r}\frac{\tau(d)m}{d^{\ell/2-2}} \ll \sum_{r\leqslant HT}\frac{1}{r} \ll \log(HT) \end{aligned}$$

and

$$\begin{aligned} S_5(T,H) &:= \sum_{d\leqslant T}\frac{\tau(d)}{d^{\ell/2-1}}\sum_{m\leqslant (T/d)^{1/2}}\sum_{n\leqslant T}\frac{\tau_3(n)}{n^{\ell/4-1/2}}\sum_{\substack{h_1,h_2\leqslant H\\ h_1 n \neq h_2 dm}}\frac{dmn}{h_1 h_2 |h_1 n - h_2 dm|} \\ &= \sum_{\substack{r_1,r_2\leqslant HT\\ r_1\neq r_2}}\frac{1}{r_1 r_2 |r_1 - r_2|}\sum_{\substack{h_2\leqslant H, d\leqslant T, m\leqslant (T/d)^{1/2}\\ h_2 dm = r_1}}\frac{\tau(d)(dm)^2}{d^{\ell/2-1}}\sum_{\substack{n\leqslant T, h_1\leqslant H\\ h_1 n = r_2}}\frac{\tau_3(n)}{n^{\ell/4-5/2}} \\ &\leqslant T\sum_{\substack{r_1,r_2\leqslant HT\\ r_1\neq r_2}}\frac{1}{r_1 r_2 |r_1 - r_2|}\sum_{h_2 dm = r_1}\frac{\tau(d)}{d^{\ell/2-2}}\sum_{h_1 n = r_2}\frac{\tau_3(n)}{n^{\ell/4-5/2}} \\ &\ll T\sum_{|r|\leqslant HT}\frac{1}{|r|}\sum_{r_2\leqslant HT}\frac{1}{r_2} \\ &\ll T(\log HT)^2. \end{aligned}$$

This proves (3.3) with the choice of $H = T$.

4. Proof of Theorem 2

For each $r \geqslant 2$, let δ_r and δ_r^* denote the infimum of $\sigma > 0$ such that

$$\int_{-\infty}^{\infty}\frac{|\zeta(\sigma+\mathrm{i}t)|^r}{|\sigma+\mathrm{i}t|^2}\,\mathrm{d}t \ll 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty}\frac{|Z_Q(\sigma+\mathrm{i}t)|^r}{|\sigma+\mathrm{i}t|^2}\,\mathrm{d}t \ll 1,$$

respectively. According to [5, Lemma 13.1], we have

$$\beta_k = \delta_{2k}. \tag{4.1}$$

On the other hand, following the proof of this lemma word by word by replacing $\zeta(s)$ by $Z_Q(s)$ and $\Delta_k(x)$ by $\Delta_k^*(Q,x)$ respectively, we can prove

$$\beta_k^* + \ell/2 - 1 = \delta_{2k}^*. \tag{4.2}$$

Finally it is easy to see that

$$|\zeta(s-\ell/2+1)| \ll |Z_Q(s)| \ll |\zeta(s-\ell/2+1)|$$

for $\ell/2-1 \leqslant \sigma \leqslant \ell/2$. Thus

$$\delta_r^* = \ell/2 - 1 + \delta_r. \tag{4.3}$$

Now Theorem 2 follows from (4.1), (4.2) and (4.3) by noting that the Lindelöf hypothesis implies $\delta_r = 1/2 - 1/r$ for any $r \geqslant 2$.

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